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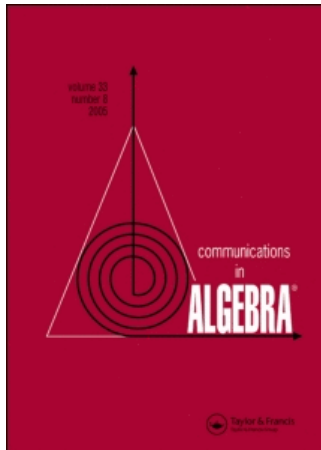
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ON THE U -INVARIANT OF P -ADIC FUNCTION FIELDS[#]

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We show that a quadratic form defined over the rational function field $\mathbb{Q}(x_1, \dots, x_n)$ of dimension at least $4 \cdot 2^n + 1$ is isotropic over all fields $\mathbb{Q}_p(x_1, \dots, x_n)$, except for finitely many primes. Partial results concerning the u -invariant of p -adic function fields are also shown.

Key Words: Function fields; Model theory; Quadratic forms.

Mathematics Subject Classification: Primary 03C60, 12L12; Secondary 12F20, 12E30.

1. INTRODUCTION

Let k be a non-real field. The u -invariant of the field k is defined to be the maximal natural number n for which there exists a quadratic form q over k , of dimension n , which is anisotropic over k . If such a natural number does not exist, the u -invariant of k is said to be infinite. It is an important open question whether the u -invariant of a finitely generated extension of a field with finite u -invariant is itself finite. For algebraic extensions a positive answer to this problem was given by Leep (1984). More precisely he proved the following: let k_1 be a finite algebraic extension of k of degree n , then we have the following upper bound for $u(k_1)$:

$$u(k_1) \leq \frac{n+1}{2} u(k).$$

In particular, if $u(k)$ is finite, then so is $u(k_1)$.

To answer the question raised above, it suffices to concentrate on purely transcendental extensions. A general result, such as Leep's for the behaviour of the u -invariant under finite transcendental extensions, is unknown so far. For certain classes of fields, satisfying stronger assumptions than the finiteness of the u -invariant results concerning the behaviour of the u -invariant under transcendental extensions have been obtained. For example, if k is a C_i -field, then by Tsen-Lang theory, $k(x)$ is a C_{i+1} -field and hence $u(k(x)) \leq 2^{i+1}$.

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One may wonder what happens with fields that are almost C_i , like for example p -adic fields. Only recently has it been shown that the u -invariant of a finitely generated field extension of transcendence degree 1 over a p -adic field is finite. More precisely, for such a field K it was shown that $u(K) \leq 10$ (see Hoffmann and Van Geel, 1998 and Parimala and Suresh, 1998). The conjectured value of the u -invariant of such a field is 8. For finitely generated field extensions over p -adic fields of higher transcendence degree, nothing is known concerning the u -invariant.

In this article we examine the u -invariant of p -adic function fields using methods from mathematical logic. Before we state our main result, we introduce some notations. Let $K = k(x_1, \dots, x_n)$ be a finitely generated, purely transcendental extension of the field k , $\text{char}(k) \neq 2$. For any natural number d , we define the set \mathcal{M}_d as follows: the set \mathcal{M}_d consists of all quadratic forms q over K such that q is isometric to a diagonal quadratic form q' , whose coefficients are polynomials in $k[x_1, \dots, x_n]$ of total degree bounded by d . Note that for each quadratic form q over K there exists a d , such that $q \in \mathcal{M}_d(K)$. We define the invariant $\mu_d(K)$ as follows:

$$\mu_d(K) = \sup\{\dim(q) : q \in \mathcal{M}_d(K) \text{ and } q \text{ is anisotropic over } K\}.$$

It is easy to verify that $\mu_d(K) \leq \mu_{d+1}(K)$ and $u(K) = \sup_d \mu_d(K)$. Our main result is the following:

Main Theorem. *For any positive integers n, d there exists a finite set of primes $P(n, d)$, such that $\mu_d(\mathbb{Q}_p(x_1, \dots, x_n)) \leq 4 \cdot 2^n$ for all prime numbers p , $p \notin P(n, d)$. If $d > n$ we have $\mu_d(\mathbb{Q}_p(x_1, \dots, x_n)) = 4 \cdot 2^n$, for all p , $p \notin P(n, d)$.*

As already observed, the conjectured value of $u(\mathbb{Q}_p(x)) = 8$, for all primes p . If however this conjecture is false, i.e., if there exist anisotropic forms of dimension 9 over all $\mathbb{Q}_p(x)$, the Main Theorem implies the non-existence of uniform bounds (i.e., uniform in p) for the degrees of the coefficients of these anisotropic forms. More precisely if there exists an anisotropic form of dimension 9 over $\mathbb{Q}_p(x)$ for every prime p , then for every p there exists an anisotropic form with coefficients in $\mathbb{Q}_p[x]$. For each p , let $q_p = \langle a_{1p}, \dots, a_{9p} \rangle$ be such an anisotropic form with polynomial coefficients such that $N(p) = \max\{\deg(a_i)\}$ is minimal. Then the Main theorem implies that there is no bound, as p varies for $N(p)$. This seems rather unlikely (for eight-dimensional anisotropic forms the bound for $N(p)$ is 1). Our Main Theorem can thus be viewed as evidence for the fact that $u(\mathbb{Q}_p(x)) = 8$. Unfortunately, our theorem does not imply that the u -invariant of almost all p -adic function fields is finite. Indeed, for a fixed integer n , it cannot be known how the set of exceptional primes $P(n, d)$ varies with d , and it could thus happen that $\bigcup_d P(n, d)$ is the set of all primes. But we do obtain the following corollary:

Corollary. *Let q be a quadratic form over $\mathbb{Q}(x_1, \dots, x_n)$ of dimension at least $4 \cdot 2^n + 1$. Then q is isotropic over $\mathbb{Q}_p(x_1, \dots, x_n)$ for all but finitely many primes.¹*

This is straightforward from the Main Theorem.

¹A similar statement also follows from a theorem by Greenleaf (1965).

The article is organized as follows. In section 2, we collect the preliminaries needed for the proof of the Main Theorem. To make the article as self-contained as possible, we have added two short sections on quadratic forms and mathematical logic. No prior knowledge of mathematical logic, nor quadratic forms is required, though some familiarity with both subjects would be helpful. In section 3, we present the proof of the Main Theorem, which is based on the Ax-Kochen-Ersov transfer theorem (AKE theorem) from the model theory of valued fields. In applying the AKE theorem we will have to express the fact that certain classes of quadratic forms over $k(x_1, \dots, x_n)$ are isotropic over $k(x_1, \dots, x_n)$ by first-order formulas over k . Hence we will need to bound the degrees of possible isotropy vectors. This seems to be impossible to do for arbitrary fields, but works over the fields we are working with. This is done in Lemmas 3.1 and 3.2.

2. PRELIMINARIES

2.1. Quadratic forms. We briefly review some basic facts about quadratic forms. For more information, we refer to Chapter 1 of Pfister (1995). A quadratic form $q(y_1, \dots, y_n)$ over k in the variables y_1, \dots, y_n is a homogenous polynomial of degree 2 in the variables. To each quadratic form $q = \sum_{i,j=1}^n a_{ij}y_iy_j$, one can associate a symmetric $n \times n$ -matrix $Q = (q_{ij})_{i,j}$ defined by $q_{ij} = (a_{ij} + a_{ji})/2$ (i.e. $q = YQY^t$, where $Y = (y_1, \dots, y_n)$). Two quadratic forms $q_1 = YQ_1Y^t$ and $q_2 = YQ_2Y^t$ are called isometric (over k) if there exists an invertible matrix A (over k) such that $Q_1 = A^tQ_2A$. If the characteristic of k is different from 2, then it is easy to see that any quadratic form is isometric to a diagonal quadratic form, i.e., a quadratic form whose associated matrix is diagonal. When a quadratic form is in diagonal form it is written as $q = \langle a_1, \dots, a_n \rangle$, where a_1, \dots, a_n are the entries on the diagonal of the associated diagonal matrix. We will always assume that a diagonal form is non-degenerate, i.e., $a_1, a_2, \dots, a_n \neq 0$. The dimension of q is defined as the rank of its associated matrix, i.e., if $q = \langle a_1, \dots, a_n \rangle$ is a non-degenerate diagonal quadratic form, its dimension is n . If $q_1 = \langle a_1, \dots, a_n \rangle$ and $q_2 = \langle b_1, \dots, b_m \rangle$ are two diagonal quadratic forms defined over k , then their orthogonal sum, denoted by $q_1 \oplus q_2$ is the quadratic form $\langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$. Similarly, if α is an element of a field containing k , then the quadratic form αq_1 is defined as $\langle \alpha a_1, \dots, \alpha a_n \rangle$. Let K be an extension field of k , a quadratic form $q = \langle a_1, \dots, a_n \rangle$ over k of dimension n is said to be isotropic over K if there exists a nonzero vector $v \in K^n$ such that $a_1v_1^2 + \dots + a_nv_n^2 = 0$. We call v an isotropy vector for q . If q is not isotropic over K it is said to be anisotropic over K . Note that if q_1 and q_2 are isometric quadratic forms over k , then q_1 is isotropic over K if and only if q_2 is.

2.2. The u -invariant. The u -invariant of a non-real field k is defined by

$$u(k) = \sup\{\dim(q) : q \text{ is a quadratic form which is anisotropic over } k\}$$

2.3. First-order field formulas. We now need to define first-order formulas. We will do this in the case of the language of fields only. In this case a first-order

formula is an expression of the form

$$\begin{aligned} Q_1 x_1, \dots, Q_n x_n : \bigvee_i \left(\bigwedge_j^{l_i} f_{ij}(x_1, \dots, x_n, y_1, \dots, y_m) \right. \\ \left. = 0 \wedge g_i(x_1, \dots, x_n, y_1, \dots, y_m) \neq 0 \right), \end{aligned}$$

where each of the f_{ij} and g_i is a polynomial with integer coefficients. Each of the Q_i is either a universal or an existential quantifier. The variables y_i are called free variables.

A formula with no free variables is called a sentence. Given a first-order sentence ϕ , it should be clear what is meant by the expression that the sentence ϕ is true in a field k (notation $k \models \phi$). For example consider the following sentences:

- (a) $\phi_1 : \forall x_1 \exists x_2 : x_1 = x_2^2 \vee x_1 = -x_2^2$
- (b) $\phi_2 : \forall x_1, x_2 \exists x_3 : x_3^2 + x_1 x_3 + x_2 x_3 = 0$
- (c) $\phi_3 : \exists x_1 \forall x_2 : x_2^p - x_2 = x_1$ (p a prime number)

Sentence ϕ_1 says that every field element is a square or the negative of a square, hence $\mathbb{R} \models \phi_1$ but ϕ_1 is not true in \mathbb{Q} . Similarly, ϕ_2 expresses that every monic quadratic polynomial has a root in the field, hence $\mathbb{C} \models \phi_2$ but ϕ_2 is not true in \mathbb{R} . The sentence ϕ_3 expresses that the map $x^p - x$ is constant, thus $k \models \phi_2$ if and only if $k = \mathbb{F}_p$.

For a more detailed exposition of these matters we refer the reader to any textbook on mathematical logic or model theory (e.g. Chapter 1 in Marker, 2002).

2.4. The AKE transfer theorem. (cf. Ax and Kochen, 1965). We now present our basic model-theoretic tool. By $\mathbb{F}_p((t))$ we denote the field of formal Laurent-series over the finite field \mathbb{F}_p , while \mathbb{Q}_p denotes the field of p -adic numbers. Let $(K_n)_n$ be a collection of fields. We say that a first-order sentence is true in almost all fields K_n if $K_n \models \phi$ for all but finitely many n .

The AKE Theorem. *Let ϕ be a first-order sentence in the language of fields. Then ϕ is true in almost all \mathbb{Q}_p if and only if it is true in almost all $\mathbb{F}_p((t))$.*

In particular, if a first-order statement is true in all $\mathbb{F}_p((t))$, then it is true in all but finitely many \mathbb{Q}_p . The finite set of primes for which the given first-order sentence is not true in \mathbb{Q}_p , is called the exceptional set of primes and it depends on the sentence in question.

We also need the following combinatorial result:

2.5. Lemma. *Let n and d be positive integers. Denote by $E(n, d)$ the number of different monomials in n variables of degree exactly d and by $B(n, d)$ the number of different monomials in n variables of degree at most d . Then we have*

$$E(n, d) = \binom{n+d-1}{d} \quad \text{and} \quad B(n, d) = \frac{(d+1)(d+2) \cdots (d+n)}{n!}$$

Proof. We first prove the formula for $E(n, d)$, by induction on the number of variables n . If $n = 1$, this is trivial. For $n > 1$ we have:

$$E(n, d) = \sum_{i=0}^d E(n-1, i).$$

Hence, using the hypothesis of induction, it suffices to show that

$$\sum_{i=0}^d \binom{n+i-2}{i} = \binom{n+d-1}{d}.$$

This can easily be done by induction on d .

To prove the formula for $B(n, d)$ we use induction on the degree d . For $d = 1$ the result is trivial. for $d > 1$ we have:

$$B(n, d) = B(n, d-1) + E(n, d).$$

Using the induction hypothesis and the formula for $E(n, d)$ we get:

$$B(n, d) = \frac{1}{n!} \prod_{i=0}^{n-1} (d+i) + \binom{n+d-1}{d},$$

which proves the result. □

3. PROOF OF THE MAIN THEOREM

We start with the following lemma, which allows us to bound, for a given quadratic form over $\mathbb{F}_p((t))(x_1, \dots, x_n)$, the degree of an isotropy vector over $\mathbb{F}_p((t))(x_1, \dots, x_n)$ in terms of the degrees of the coefficients of the given form. Recall that for a rational function field $k(x_1, \dots, x_n)$ the set $\mathcal{M}_d(k(x_1, \dots, x_n))$ was defined as the set consisting of all quadratic forms q over $k(x_1, \dots, x_n)$ such that q is isometric to a diagonal quadratic form q' whose coefficients are polynomials in $k[x_1, \dots, x_n]$ of total degree bounded by d .

3.1. Lemma. *Let n, m and d be positive integers and suppose that $m > 4.2^n$. Then there exists a positive integer $b(n, d)$ such that for any prime number $p \neq 2$ the following holds: every quadratic form of dimension m which is contained in $\mathcal{M}_d(\mathbb{F}_p((t))(x_1, \dots, x_n))$ has an isotropy vector $v = (v_1, \dots, v_m)$ such that $v_i \in \mathbb{F}_p((t))[x_1, \dots, x_n]$ and $\deg(v_i) \leq b(n, d)$.*

Proof. Let q be a form of dimension m given by

$$q(Y_1, \dots, Y_m) = \sum_{i,j=1, i \leq j}^m a_{ij} Y_i Y_j,$$

where each coefficient a_{ij} is a polynomial of degree at most d . Write each Y_i as a polynomial of degree N with indeterminate coefficients $y_{i\ell}$ in $\mathbb{F}_p((t))$ (where ℓ is a

multi-index (ℓ_1, \dots, ℓ_n) ranging over \mathbb{N}^n with $\ell_1 + \dots + \ell_n = N$. Substituting these expressions for Y_i in q results in a polynomial in the variables x_1, \dots, x_n whose coefficients q_i are quadratic forms in the y_{ij} . The quadratic form q will have a non-trivial zero over $\mathbb{F}_p((t))[x_1, \dots, x_n]$ of degree bounded by N if and only if the quadratic forms q_i have a common non-trivial zero in $\mathbb{F}_p((t))$. The latter system is a system of

$$\frac{(2N + d + 1) \cdots (2N + d + n)}{n!}$$

quadratic forms in

$$m \left(\frac{(N + 1) \cdots (N + n)}{n!} \right)$$

variables. Since $\mathbb{F}_p((t))$ is a C_2 -field, this system will have a solution provided

$$m \left(\frac{(N + 1) \cdots (N + n)}{n!} \right) > 4 \frac{(2N + d + 1) \cdots (2N + d + n)}{n!} \quad (1)$$

which is equivalent to

$$m(N + 1) \cdots (N + n) > 4(2N + d + 1) \cdots (2N + d + n).$$

Viewing both sides of the inequality as polynomials in N , observe that the left-hand side is a polynomial of degree n with highest degree coefficient m , while the right-hand side is a polynomial of degree n with highest degree coefficient 2^{n+2} . By assumption, $m > 2^{n+2}$ and hence the inequality (1) will be satisfied for some value of N , say $N(m, d)$ which is large enough. It is easy to see that we can take $b(n, d)$ equal to $N(2^{n+2} + 1, d)$. \square

We also need the following definability result:

3.2. Lemma. *Let $\mathbf{m} = (m_1, m_2, m_3, m_4)$ be a tuple of positive integers. There exists a first order sentence $\phi_{\mathbf{m}}$ in the language of fields such that for every field k of characteristic different from 2 the following holds: every quadratic form q in $\mathcal{M}_{m_2}(k(x_1, \dots, x_{m_1}))$ of dimension m_3 has an isotropy vector of degree bounded by m_4 if and only if $k \models \phi_{\mathbf{m}}$.*

Proof. We indicate briefly how one obtains the formula $\phi_{\mathbf{m}}$ and leave the details to the reader. By the definition of $\mathcal{M}_{m_2}(k(x_1, \dots, x_{m_1}))$, $\phi_{\mathbf{m}}$ should be true in k , just in the case when every non-degenerate diagonal form q of dimension m_3 with coefficients from $k[x_1, \dots, x_{m_1}]$ of total degree at most m_2 has an isotropy vector of total degree bounded by m_4 . This condition would be true just in the case for all non-zero polynomials a_1, \dots, a_{m_3} in $k[x_1, \dots, x_{m_1}]$ of total degree at most m_2 ,

there are polynomial y_1, \dots, y_{m_3} of total degree at most m_4 which yield a non-trivial zero of the diagonal form $\langle a_1, \dots, a_{m_3} \rangle$, i.e.:

$$\forall a_1, \dots, a_{m_3} \exists y_1, \dots, y_{m_3} : \\ (\bigvee_i a_i = 0) \vee ((\bigwedge_i \deg(a_i) \leq m_2) \wedge (\bigwedge_i \deg(y_i) \leq m_4) \wedge (\bigvee_i y_i \neq 0) \wedge (\sum_i a_i y_i^2 = 0)).$$

Since the degrees of the polynomials over which we quantify are bounded, we can replace these quantifiers, which range over the polynomial ring, by quantifiers ranging over the ground field k . As in the proof of the previous lemma, the quantifier-free part of the formula, which still involves the indeterminates x_1, \dots, x_{m_2} can easily be replaced by a quantifier free formula over the base field k . This yields the formula $\phi_{\mathbf{m}}$. \square

3.3. Proof of the Main Theorem. Fix positive integers n and d . By 3.1 and 3.2 we have that for any prime number $p \neq 2$, $\mathbb{F}_p((t))$ satisfies the sentence $\psi = \phi_{\mathbf{m}}$, with $\mathbf{m} = (n, d, 4 \cdot 2^n + 1, b(n, d))$. By the AEK transfer theorem, there exists a finite set of primes $P(n, d)$ such that for all primes $p \notin P(n, d)$, \mathbb{Q}_p satisfies ψ . This implies that any quadratic form in $\mathcal{M}_d(\mathbb{Q}_p(x_1, \dots, x_n))$, $p \notin P(n, d)$, of dimension $4 \cdot 2^n + 1$ is isotropic and hence any quadratic form of dimension at least $4 \cdot 2^n + 1$ is isotropic. Hence $\mu_d(\mathbb{Q}_p(x_1, \dots, x_n)) \leq 4 \cdot 2^n$ for all but finitely many primes.

We now prove the equality. If q is an anisotropic quadratic form over a field k of dimension n , then the $2n$ -dimensional quadratic form $q' := q \oplus xq$, with x transcendental over k , is anisotropic over $k(x)$. Indeed, suppose that q' is isotropic over $k(x)$, then without loss of generality we may assume that it has an isotropy vector $v = (v_1, v_2)$, $v_1, v_2 \in k[x]^n$. Then $q(v_1)$ has even degree in x (if not, q would be isotropic over k) and $xq(v_2)$ has odd degree in x . This implies that q is isotropic over $k(x)$ and hence over k , which contradicts the assumption. It is now easy to see that if k admits an anisotropic quadratic form of dimension ℓ , then for $d \geq n$, $\mathcal{M}_d(k(x_1, \dots, x_n))$ contains an anisotropic quadratic form of dimension $2^n \ell$. Since \mathbb{Q}_p admits an anisotropic form of dimension 4, we get that for $d \geq n$, $\mathcal{M}_d(\mathbb{Q}_p(x_1, \dots, x_n))$ contains an anisotropic quadratic form of dimension $4 \cdot 2^n$, and hence $\mu_d(\mathbb{Q}_p(x_1, \dots, x_n)) \geq 4 \cdot 2^n$.

3.4. Remark. Both 3.1 and 3.2 remain true in case the fields are of characteristic 2 (after suitably modifying the definition of \mathcal{M}_d). Since in that case we have to take into account the fact that not all quadratic forms are diagonal, the proofs just become a little more complicated to write. Note that even if we would have taken the trouble of incorporating the characteristic 2 case, we wouldn't have obtained a stronger theorem.

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REFERENCES

- Ax, J., Kochen, S. (1965). Diophantine problems over local fields I. *Amer. J. Math.* 87:605–630.
- Greenleaf, M. (1965). Irreducible subvarieties and rational points. *Amer. J. Math.* 87:25–31.
- Hoffmann, D., Van Geel, J. (1998). Zeros and norm groups of quadratic forms over function fields in one variable over a local non-dyadic field. *J. Ramanujan Math. Soc.* 13:85–110.
- Leep, D. (1984). Systems of quadratic forms. *J. Reine Angew. Math.* 350:109–116.
- Marker, D. (2002). *An Introduction to Model Theory*. Graduate texts in Mathematics 217. New-York: Springer Verlag.
- Parimala, R., Suresh, V. (1998). Isotropy of quadratic forms over function fields of p -adic curves. *Inst. Hautes Etudes Sci. Publ. Math.* 88:129–150.
- Pfister, A. (1995). *Quadratic Forms with Applications to Algebraic Geometry and Topology*. London Mathematical Society Lecture Note Series 217. Cambridge: Cambridge University Press.